



On \bar{h} -Jacobsthal and \bar{h} -Jacobsthal–Lucas sequences, and related quaternions

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Abstract

In this paper, inspired by recent articles of A. Szynal-Liana & I. Wloch and F. T. Aydin & S. Yüce (see [26] and [2]), we will introduce the \bar{h} -Jacobsthal quaternions and the \bar{h} -Jacobsthal–Lucas sequences and their associated quaternions. The new results that we have obtained extend most of those obtained in [26].

1 Introduction

Quaternions are objects introduced by Hamilton order to attempt a definition of coordinate system different from cartesian one. Hamilton idea's had a great success by his contemporary scientist, and in particular it had a strong application in physics. Actually, after a period of shadow, they have been recently considered in different branches of mathematics (see [23] and reference therein), and many researches are devoted to them.

The set of real quaternions is denoted by \mathbb{H} and a quaternion number appears in the form $a + bi + cj + dk$, where $a, b, c, d \in \mathbb{R}$. The basic rules is given by

$$i^2 = j^2 = k^2 = ijk = -1. \quad (1)$$

For an introduction to quaternions theory see [29] (see also the introduction of [26] for basic rules).

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In 1963 A. F. Horadam had the idea to investigate special quaternion numbers. Precisely, Fibonacci quaternions FQ_n , that are quaternions of the form $FQ_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}$ [18]. Following him other authors had investigated other kind of quaternions, and many interesting properties of Fibonacci and Lucas quaternions had ben obtained (see [14], [22]).

After, S.K. Nurkan and İ.A Gven in [24] defined dual Fibonacci quaternions and dual Lucas quaternions. Overcoming a common point of view, in [13, 15] Halici investigated complex Fibonacci quaternions, and very recently İpek [21] have investigated (p, q) -Fibonacci quaternions.

In [20] Horadam mentioned the possibility of introducing Pell quaternions and generalized Pell quaternions. Interesting results of Pell quaternions, Pell–Lucas quaternions and Tribonacci quaternions obtained recently can be found in [8, 9], [25] and [7].

Also we mention that investigations on generalized Jacobsthal and Jacobsthal–Lucas polynomials, $J_n(x)$ and $j_n(x)$, can be found in [5] and [6].

Recently, in [2] and [26] the authors starting from Jacobsthal sequence $(J_n)_n$ and Jacobsthal–Lucas sequence $(j_n)_n$

$$\begin{cases} J_0 = 0, J_1 = 1 \\ J_n = J_{n-1} + 2J_{n-2}, \end{cases} \quad \begin{cases} j_0 = 2, j_1 = 1 \\ j_n = j_{n-1} + 2j_{n-2}, \end{cases} \quad (2)$$

considered the Jacobsthal quaternions and Jacobsthal–Lucas quaternions, obtaining properties and matrix representation of such numbers.

As the author had highlighted in [26], the numbers of Fibonacci type have many applications in distinct area of mathematics (see also [1], [28], [11] and reference therein). Therefore, it seems a natural question to investigate quaternions connected to either a Fibonacci-like sequence (see [28]) or more generally, to a whichever recursive sequence of second order; but it seems not easy to develop a general theory for quaternions connected to a whichever sequence too much different from Fibonacci or Lucas sequences.

In this paper in order to attempt to this kind of investigation, we restrict our attention to quaternions connected to generalizations of Jacobsthal sequencens $(J_{h,n})_n$ and Jacobsthal–Lucas sequence (see Equations 3).

Let h be a complex number, we will define \bar{h} -Jacobsthal sequence $(J_{h,n})_n$ and \bar{h} -Jacobsthal–Lucas sequence $(j_{h,n})_n$ the homogenous recursive sequences of second order with constant coefficients, whose characteristic polynomial is $x^2 - (h - 1)x - h = (x - h)(x + 1)$, and initial conditions are 0, 1 and 2, 1 respectively:

$$\begin{cases} J_{h,0} = 0, J_{h,1} = 1 \\ J_{h,n} = (h-1)J_{h,n-1} + hJ_{h,n-2} \end{cases} \quad \text{and} \quad \begin{cases} j_{h,0} = 2, j_{h,1} = 1 \\ j_{h,n} = (h-1)j_{h,n-1} + hj_{h,n-2}. \end{cases} \quad (3)$$

Note that the above families of sequences are different from those considered in [27] (see Definition 2.1 on page 470), in [3] (see Definition (1) on page 490), and in [4] (Definition (1)-(2) on page 38); moreover, it is clear that when $h = 2$ we have the quoted Jacobsthal sequence and Jacobsthal–Lucas sequence.

Taking the paper of A. Szynal-Liana and I. Włoch as model we will extend most of the results showed in [26] to \bar{h} -Jacobsthal quaternions and \bar{h} -Jacobsthal–Lucas quaternions.

2 Basic properties

In [19] the author provided many basic identities for Jacobsthal numbers, (J_n) , and Jacobsthal–Lucas numbers, (j_n) (see also [26, Equations 6–14]):

- i) $j_{n+1} + j_n = 3(J_{n+1} + J_n) = 3 \cdot 2^n$,
- ii) $j_{n+1} - j_n = 3(J_{n+1} - J_n) + 4(-1)^{n+1} = 2^n + 2(-1)^{n+1}$,
- iii) $j_{n+r} + j_{n-r} = 3(J_{n+r} + J_{n-r}) + 4(-1)^{n-r} = 2^{n-r}(2^{2r} + 1) + 2(-1)^{n-r}$,
- iv) $j_{n+r} - j_{n-r} = 3(J_{n+r} - J_{n-r}) = 2^{n-r}(2^{2r} - 1)$,
- v) $J_n + j_n = 2J_{n+1}$,
- vi) $3J_n + j_n = 2^{n+1}$,
- vii) $j_n J_n = J_{2n}$,
- viii) $J_m j_n + J_n j_m = 2J_{m+n}$,
- ix) $J_m j_n - J_n j_m = (-1)^n 2^{n+1} J_{m-n}$.

Remark 2.1. We note that when $h = -1$, then the characteristic polynomial has the unique root -1 (with multiplicity 2). In this case the sequences $(J_{h,n})_n$ and $(j_{h,n})_n$ are both geometric and their study can be considered trivial.

Let $h \neq -1$ be a complex number. In order to extend similar properties to \bar{h} -Jacobsthal and \bar{h} -Jacobsthal–Lucas sequences we will provide the Binet’s formula for these sequences.

Lemma 2.1. [*Binet's Formula*] Let $h \neq -1$ be a complex number, and let $(J_{h,n})_n$ be the \bar{h} -Jacobsthal sequence and $(j_{h,n})_n$ be the \bar{h} -Jacobsthal-Lucas sequence. Then the Binet's formulas reads:

$$J_{h,n} = \frac{1}{h+1} [h^n - (-1)^n] \quad \text{and} \quad (4)$$

$$j_{h,n} = \frac{1}{h+1} [3h^n + (2h-1)(-1)^n]. \quad (5)$$

Proof. The characteristic polynomial of both the sequences $(J_{h,n})_n$ and $(j_{h,n})_n$ is

$$x^2 - (h-1)x - h = (x-h)(x+1), \quad (6)$$

and by hypothesis it has two distinct roots, namely -1 and h . It follows that there exist suitable $c_1, c_2, d_1, d_2 \in \mathbb{C}$ such that the Binet's formula for $(J_{h,n})_n$ and $(j_{h,n})_n$ is of the type:

$$J_{h,n} = c_1 h^n + c_2 (-1)^n \quad \text{and} \quad j_{h,n} = d_1 h^n + d_2 (-1)^n, \quad \text{respectively.} \quad (7)$$

In particular, using the initial conditions we have the relations:

$$\begin{cases} J_{h,0} = c_1 h^0 + c_2 (-1)^0 = 0 \\ J_{h,1} = c_1 h^1 + c_2 (-1)^1 = 1 \end{cases} \quad \text{and} \quad \begin{cases} j_{h,0} = d_1 h^0 + d_2 (-1)^0 = 2 \\ j_{h,1} = d_1 h^1 + d_2 (-1)^1 = 1 \end{cases}.$$

Thus we have the following systems

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 h - c_2 = 1 \end{cases} \quad \text{and} \quad \begin{cases} d_1 + d_2 = 2 \\ d_1 h - d_2 = 1 \end{cases},$$

whose solutions are:

$$\begin{cases} c_1 = \frac{1}{h+1} \\ c_2 = -\frac{1}{h+1} \end{cases} \quad \text{and} \quad \begin{cases} d_1 = \frac{3}{h+1} \\ d_2 = \frac{2h-1}{h+1} \end{cases}.$$

□

Remark 2.2. We highlight that when $h \geq 0$ the above Binet's formula 4 can be directly detected from a more general Binet's Formula for Horadam sequences given by Halici (see [13, Equation 2.5]); on the other hand, we note that for the general Horadam sequences of second order, the cited formula do not hold when the roots of characteristic polynomials coincides.

Remark 2.3. We highlight that when $h = 2$ the above Binet's formulas 4 and 5 give the Binet's Formula of the Jacobstal sequence $(J_n)_n$:

$$J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 3, J_4 = 5, J_5 = 11, J_6 = 21, \dots$$

and Jacobstal-Lucas sequence $(j_n)_n$:

$$j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 7, j_4 = 17, j_5 = 31, j_6 = 65, \dots$$

Precisely, the Binet's formulas for J_n and j_n are:

$$J_n = J_{2,n} = \frac{1}{2+1} [2^n - (-1)^n] \quad (\text{see [2, Equation 5]}), \quad (8)$$

$$j_n = j_{2,n} = \frac{1}{2+1} [3 \cdot 2^n + (2 \cdot 2 - 1)(-1)^n] = 2^n + (-1)^n \quad (\text{see [2, Equation 6]}). \quad (9)$$

Note that in the last equation the coefficients of the Binet's Formula are both "+1".

Next result show that each term of $(j_{h,n})_n$ has one more representation. It will be useful for proving condition (vi) of Lemma 2.3.

Corollary 2.2. *Let $h \neq -1$ be a complex number, and let $(J_{h,n})_n$ be the \bar{h} -Jacobsthal sequence and $(j_{h,n})_n$ be the \bar{h} -Jacobsthal-Lucas sequence. Then the following identities hold:*

$$i) \quad j_{h,n} = 3[h^{n-1} + (-1)h^{n-2} + \dots + (-1)^{n-2}h] - (-1)^n.$$

$$ii) \quad j_{h,n} = 3J_{h,n} + 2(-1)^n.$$

Proof. i) By Equation 5, for every integer n we have:

$$\begin{aligned} j_{h,n} &= \frac{1}{h+1} [3h^n + (2h-1)(-1)^n] = \frac{1}{h+1} [3h^n + 2h(-1)^n - (-1)^n] \\ &= \frac{1}{h+1} [3(h^n - (-1)^n) + 2(h+1)(-1)^n] \\ &= 3[h^{n-1} + (-1)h^{n-2} + \dots + (-1)^{n-2}h + (-1)^{n-1}] + 2(-1)^n \\ &= 3[h^{n-1} + (-1)h^{n-2} + \dots + (-1)^{n-2}h] + 3 \cdot (-1)^{n-1} + 2(-1)^n \\ &= 3[h^{n-1} + (-1)h^{n-2} + \dots + (-1)^{n-2}h] - 3 \cdot (-1)^n + 2(-1)^n \\ &= 3[h^{n-1} + (-1)h^{n-2} + \dots + (-1)^{n-2}h] - (-1)^n. \end{aligned}$$

ii) By Equation 5, for every integer n we have:

$$\begin{aligned}
 j_{h,n} &= \frac{1}{h+1} [3h^n + (2h-1)(-1)^n] \\
 &= \frac{1}{h+1} [3h^n - 3(-1)^n + 3(-1)^n + 2h(-1)^n - (-1)^n] \\
 &= \frac{1}{h+1} [3(h^n - (-1)^n) + 2h(-1)^n + 2(-1)^n] = 3J_{h,n} + 2(-1)^n. \quad \square
 \end{aligned}$$

The next Lemma shows that similar properties to (i)–(vii) also hold for \bar{h} -Jacobsthal and \bar{h} -Jacobsthal–Lucas sequences:

Lemma 2.3. *Let $h \neq -1$ be a complex number, and let $(J_{h,n})_n$ be the \bar{h} -Jacobsthal sequence and $(j_{h,n})_n$ be the \bar{h} -Jacobsthal–Lucas sequence. Then the following identities hold:*

- i) $j_{h,n+1} + j_{h,n} = 3(J_{h,n+1} + J_{h,n}) = 3h^n$. In particular, $(J_{h,n+1} + J_{h,n}) = h^n$.
- ii) $J_{h,n+1} - J_{h,n} = \frac{1}{h+1}(h^n(h-1) - 2(-1)^{n+1})$ and
 $j_{h,n+1} - j_{h,n} = \frac{3}{h+1}h^n(h-1) + 2(-1)^{n+1}\left(\frac{2h-1}{h+1}\right)$
 $= 3(J_{h,n+1} - J_{h,n}) - 4(-1)^n$.
- iii) $J_{h,n+r} + J_{h,n-r} = \frac{1}{h+1}[h^{n-r}(h^{2r} + 1) - 2(-1)^{n+r}]$ and
 $j_{h,n+r} + j_{h,n-r} = \frac{1}{h+1}[3h^{n-r}(h^{2r} + 1) + 2(2h-1)(-1)^{n+r}]$
 $= 3(J_{h,n+r} + J_{h,n-r}) + 4(-1)^{n+r}$.
- iv) $J_{h,n+r} - J_{h,n-r} = \frac{h^{n-r}}{h+1}(h^{2r} - 1)$ and
 $j_{n+r} - j_{n-r} = \frac{3h^{n-r}}{h+1}(h^{2r} - 1) = 3(J_{n+r} - J_{n-r})$.
- v) $J_{h,n} + j_{h,n} = \frac{2}{h+1}[2h^n + (h-1)(-1)^n]$.
- vi) $(h+1)J_{h,n} + j_{h,n} = h^n - 2(-1)^n + 3[h^{n-1} + (-1)h^{n-2} + \dots + (-1)^{n-2}h]$.
- vii) $j_{h,n}J_{h,n} = \frac{1}{(h+1)^2}[3h^{2n} + 1 + 2(h-2)h^n(-1)^n]$.

Proof. By Equations 4 and 5, we have:

$$\begin{aligned}
 \text{(i) } J_{h,n+1} + J_{h,n} &= \frac{1}{h+1} [h^{n+1} - (-1)^{n+1}] + \frac{1}{h+1} [h^n - (-1)^n] \\
 &= \frac{h^n}{h+1} (h+1) = h^n. \\
 j_{h,n+1} + j_{h,n} &= \frac{1}{h+1} [3h^{n+1} + (2h-1)(-1)^{n+1}] \\
 &\quad + \frac{1}{h+1} [3h^n + (2h-1)(-1)^n] = 3h^n. \\
 \text{(ii) } J_{h,n+1} - J_{h,n} &= \frac{1}{h+1} [h^{n+1} - (-1)^{n+1}] - \frac{1}{h+1} [h^n - (-1)^n] \\
 &= \frac{1}{h+1} (h^n(h-1) - 2(-1)^{n+1}). \\
 j_{h,n+1} - j_{h,n} &= \frac{1}{h+1} [3h^{n+1} + (2h-1)(-1)^{n+1}] \\
 &\quad - \frac{1}{h+1} [3h^n + (2h-1)(-1)^n] \\
 &= \frac{1}{h+1} [3h^n(h-1) - 2(2h-1)(-1)^n]. \text{ Moreover,} \\
 3(J_{h,n+1} - J_{h,n}) - 4(-1)^n &= \frac{3}{h+1} (h^n(h-1) + 2(-1)^n) - 4(-1)^n \\
 &= \frac{3}{h+1} h^n(h-1) + 2(-1)^n \left(\frac{3}{h+1} - 2 \right) \\
 &= \frac{3}{h+1} h^n(h-1) + 2(-1)^n \left(\frac{1-2h}{h+1} \right) = j_{h,n+1} - j_{h,n}. \\
 \text{(iii) } J_{h,n+r} + J_{h,n-r} &= \frac{1}{h+1} [h^{n+r} - (-1)^{n+r}] + \frac{1}{h+1} [h^{n-r} - (-1)^{n-r}] \\
 &= \frac{1}{h+1} (h^{n-r}(h^{2r} + 1) - 2(-1)^{n+r}). \\
 j_{h,n+r} + j_{h,n-r} &= \frac{1}{h+1} [3h^{n+r} + (2h-1)(-1)^{n+r}] \\
 &\quad + \frac{1}{h+1} [3h^{n-r} + (2h-1)(-1)^{n-r}] \\
 &= \frac{1}{h+1} [3h^{n-r}(h^{2r} + 1) + 2(2h-1)(-1)^{n+r}] \\
 &= \frac{3}{h+1} h^{n-r}(h^{2r} + 1) + \frac{1}{h+1} 2(2h-1)(-1)^{n+r}. \text{ Moreover,} \\
 3(J_{h,n+r} + J_{h,n-r}) + 4(-1)^{n+r} &= \frac{3}{h+1} h^{n-r}(h^{2r} + 1) - 2(-1)^{n+r} \left(\frac{3}{h+1} - 2 \right) \\
 &= \frac{3}{h+1} h^{n-r}(h^{2r} + 1) + 2(-1)^{n+r} \left(\frac{2h-1}{h+1} \right)
 \end{aligned}$$

$$\begin{aligned}
 &= j_{h,n+r} + j_{h,n-r}. \\
 \text{(iv) } J_{h,n+r} - J_{h,n-r} &= \frac{1}{h+1} [h^{n+r} - (-1)^{n+r}] - \frac{1}{h+1} [h^{n-r} - (-1)^{n-r}] \\
 &= \frac{h^{n-r}}{h+1} (h^{2r} - 1); \\
 j_{h,n+r} - j_{h,n-r} &= \frac{1}{h+1} [3h^{n+r} + (2h-1)(-1)^{n+r}] \\
 &\quad - \frac{1}{h+1} [3h^{n-r} + (2h-1)(-1)^{n-r}] \\
 &= \frac{3h^n}{h+1} (h^r - h^{-r}) = \frac{3h^{n-r}}{h+1} (h^{2r} - 1) = 3(J_{h,n+r} - J_{h,n-r}). \\
 \text{(v) } J_{h,n} + j_{h,n} &= \frac{1}{h+1} [h^n - (-1)^n] + \frac{1}{h+1} [3h^n + (2h-1)(-1)^n] \\
 &= \frac{2}{h+1} (2h^n + (h-1)(-1)^n). \\
 \text{(vi) Applying Equation 4 and Corollary 2.2, we have that:} \\
 (h+1)J_{h,n} + j_{h,n} &= h^n - 2(-1)^n + 3[h^{n-1} + (-1)h^{n-2} + \dots + (-1)^{n-2}h]. \\
 \text{(vii) } J_{h,n}j_{h,n} &= \frac{1}{(h+1)^2} (h^n - (-1)^n) [3h^n + (2h-1)(-1)^n] \\
 &= \frac{1}{(h+1)^2} [3h^{2n} + 1 + 2(h-2)h^n(-1)^n]. \quad \square
 \end{aligned}$$

The last part of this section is devoted to the study of the limit of the sequences of the ratios of \bar{h} -Jacobsthal, and \bar{h} -Jacobsthal Lucas sequences. Recall that if (F_n) is a recursive sequence definitively different from 0, then the sequence of ratio (F_n/F_{n-1}) may assume different behavior; precisely, it may be that it has a limit, or not. This strongly depend on the initial condition and from the characteristic polynomial (see [11] for a detailed analysis). In the positive case, following [11], we will call “ $\lim \frac{F_n}{F_{n-1}}$ ” as the *Kepler limit* of (F_n) , in honor of the great mathematician J. Kepler who first highlighted that for Fibonacci series that limit exists, and it is the golden mean Φ (see the book “*Harmonices Mundi*”, 1619, p. 273, see [17, 16]).

As highlighted in Remark 2.1, we note that if $h = -1$ then both $(J_{h,n})_n$ and $(j_{h,n})_n$ are geometric progressions with the same mean, namely “ -1 ”, so that in this case their Kepler limit exists and it is exactly “ -1 ”. The following result characterizes those \bar{h} -Jacobsthal and \bar{h} -Jacobsthal–Lucas sequences, which have the Kepler Limit.

Theorem 2.4. *Let $h \neq -1$ be a complex number, and let $(J_{h,n})_n$ be the \bar{h} -Jacobsthal sequence and $(j_{h,n})_n$ be the \bar{h} -Jacobsthal-Lucas sequence, then:*

i) $(J_{h,n})_n$ has Kepler limit if and only if $|h| \neq 1$, in which case we have

$$\lim \frac{J_{h,n}}{J_{h,n-1}} = \max\{|h|, 1\}.$$

ii) If $h \neq \frac{1}{2}$ then $(j_{h,n})_n$ has Kepler limit if and only if $|h| \neq 1$, in which case we have

$$\lim \frac{j_{h,n}}{j_{h,n-1}} = \max\{|h|, 1\}.$$

If $h = \frac{1}{2}$ then $(j_{h,n})_n$ has Kepler limit, and we have

$$\lim \frac{j_{h,n}}{j_{h,n-1}} = \frac{1}{2}.$$

Proof. i) The coefficients of the Binet's Formula of $(J_{h,n})_n$ (see Equation 4) are both different from 0; thus " $(J_{h,n})_n$ is definitively different from 0 and it has Kepler limit if and only if the characteristic polynomial has two roots of different modulus" (see [11, Theorem 2.3] for general case). By Equation 6, the characteristic polynomial of $(J_{h,n})_n$ is $x^2 - (h-1)x - h = (x-h)(x+1)$, thus $(J_{h,n})_n$ is definitively different from 0 and it has Kepler limit if and only if $|h| \neq |-1| = 1$. It is well known that in such case the Kepler limit is exactly the root of maximum modulus (see for example [12], and references therein), and this shows the condition 'i'.

ii) The coefficients of the Binet's Formula of $(j_{h,n})_n$ are $\frac{3}{h+1}$ and $\frac{2h-1}{h+1}$ (see Equation 5). Thus we split into two cases:

ii. 1) $h \neq \frac{1}{2}$. In this case the coefficients $\frac{3}{h+1}$ and $\frac{2h-1}{h+1}$ are both different from 0, and arguing as above, we have that $(j_{h,n})_n$ is definitively different from 0 and has Kepler limit if and only if $|h| \neq |-1| = 1$.

ii. 2) $h = \frac{1}{2}$. In this case the coefficient of the root -1 is 0. Thus the unique root that appears in the Binet's formula is $\frac{1}{2}$, so that $(j_{h,n})_n$ is a geometric progression and the Kepler limit is $\frac{1}{2}$. \square

3 The \bar{h} -Jacobsthal and \bar{h} -Jacobsthal–Lucas Quaternions

Let h be a real number. Following [26] we define the n -th \bar{h} -Jacobsthal quaternion $JQ_{h,n}$ and the n -th \bar{h} -Jacobsthal–Lucas quaternion $JLQ_{h,n}$ as

$$JQ_{h,n} = J_{h,n} + iJ_{h,n+1} + jJ_{h,n+2} + kJ_{h,n+3} \quad (10)$$

$$JLQ_{h,n} = j_{h,n} + ij_{h,n+1} + jj_{h,n+2} + kj_{h,n+3}. \quad (11)$$

Example 3.1. If we put $h = 2$ in the Equations 10 and 11 we have the Jacobsthal quaternions sequence (JQ_n) and the Jacobsthal–Lucas quaternions sequence (JLQ_n) studied in [26]:

$$\begin{array}{ll} JQ_0 = & 0 + i1 + j1 + k3 & JLQ_0 = & 2 + i1 + j5 + k7 \\ JQ_1 = & 1 + i1 + j3 + k5 & JLQ_1 = & 1 + i5 + j7 + k17 \\ JQ_2 = & 1 + i3 + j5 + k11 & JLQ_2 = & 5 + i7 + j17 + k31 \\ JQ_3 = & 3 + i5 + j11 + k21 & JLQ_3 = & 7 + i17 + j31 + k65 \\ JQ_4 = & 5 + i11 + j21 + k43 & JLQ_4 = & 17 + i31 + j65 + k127 \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

The following result extends Theorem 1 of [26], which can be obtained by next theorem just replacing $h = 2$.

Theorem 3.1. *Let $h \neq -1$ be a real number, and let $(JQ_{h,n})_n$ be the \bar{h} -Jacobsthal quaternions sequence. Then for every positive integers n and r , we have:*

$$\begin{array}{l} i) \quad JQ_{h,n+1} + JQ_{h,n} = h^n(1 + hi + h^2j + h^3k). \\ ii) \quad JQ_{h,n+1} - JQ_{h,n} \\ \quad \quad = \frac{1}{h+1} [(h-1)h^n(1 + hi + h^2j + h^3k) + 2(-1)^n(1 - i + j - k)]. \\ iii) \quad JQ_{h,n+r} + JQ_{h,n-r} \\ \quad \quad = \frac{1}{h+1} [h^{n-r}(h^{2r} + 1)(1 + hi + h^2j + h^3k) - 2(-1)^{n-r}(1 - i + j - k)]. \\ iv) \quad JQ_{h,n+r} - JQ_{h,n-r} = \frac{1}{h+1} h^{n-r}(h^{2r} - 1)(1 + hi + h^2j + h^3k). \\ v) \quad N(JQ_{h,n}) = \frac{1}{(h+1)^2} [h^{2n}(1 + h^2)(1 + h^4) + 2(-1)^n h^n (h-1)(1 + h^2) + 4]. \end{array}$$

Proof. (i) By definition,

$$JQ_{h,n+1} + JQ_{h,n} = (J_{h,n+1} + J_{h,n}) + i(J_{h,n+2} + J_{h,n+1}) \\ + j(J_{h,n+3} + J_{h,n+2}) + k(J_{h,n+4} + J_{h,n+3}).$$

Using the property (i) in Lemma 2.3, we have

$$JQ_{h,n+1} + JQ_{h,n} = h^n + ih^{n+1} + jh^{n+2} + kh^{n+3} = h^n(1 + hi + h^2j + h^3k).$$

(ii) By definition,

$$JQ_{h,n+1} - JQ_{h,n} = (J_{h,n+1} - J_{h,n}) + i(J_{h,n+2} - J_{h,n+1}) + \\ j(J_{h,n+3} - J_{h,n+2}) + k(J_{h,n+4} - J_{h,n+3}).$$

By the property (ii) proved in Lemma 2.3, we have

$$(h+1)(JQ_{h,n+1} - JQ_{h,n}) = (h-1) \cdot h^n - 2(-1)^{n+1} \\ + i((h-1) \cdot h^{n+1} - 2(-1)^{n+2}) \\ + j((h-1) \cdot h^{n+2} - 2(-1)^{n+3}) \\ + k((h-1) \cdot h^{n+3} - 2(-1)^{n+4}) \\ = (h-1) \cdot h^n - 2(-1)^{n-1} \\ + i((h-1) \cdot h^{n+1} + 2(-1)^{n-1}) \\ + j((h-1) \cdot h^{n+2} - 2(-1)^{n-1}) \\ + k((h-1) \cdot h^{n+3} + 2(-1)^{n-1}) \\ = (h-1)h^n(1 + hi + h^2j + h^3k) - 2(-1)^{n-1}(1 - i + j - k).$$

(iii) By definition,

$$JQ_{h,n+r} + JQ_{h,n-r} = (J_{h,n+r} + J_{h,n-r}) + i(J_{h,n+r+1} + J_{h,n-r+1}) \\ + j(J_{h,n+r+2} + J_{h,n-r+2}) + k(J_{h,n+r+3} + J_{h,n-r+3}).$$

By property (iii) proved in Lemma 2.3, we have

$$(h+1)(JQ_{h,n+r} + JQ_{h,n-r}) = h^{n-r}(h^{2r} + 1) - 2(-1)^{n-r} \\ + i(h^{n+1-r}(h^{2r} + 1) - 2(-1)^{n+1-r}) \\ + j(h^{n+2-r}(h^{2r} + 1) - 2(-1)^{n+2-r}) \\ + k(h^{n+3-r}(h^{2r} + 1) - 2(-1)^{n+3-r}) \\ = h^{n-r}(h^{2r} + 1) + 2(-1)^{n-1-r} \\ + i(h^{n+1-r}(h^{2r} + 1) - 2(-1)^{n-1-r}) \\ + j(h^{n+2-r}(h^{2r} + 1) + 2(-1)^{n-1-r}) \\ + k(h^{n+3-r}(h^{2r} + 1) - 2(-1)^{n-1-r}) \\ = h^{n-r}(h^{2r} + 1)(1 + hi + h^2j + h^3k) \\ + 2(-1)^{n-1-r}(1 - i + j - k).$$

(iv) By definition,

$$JQ_{h,n+r} - JQ_{h,n-r} = (J_{h,n+r} - J_{h,n-r}) + i(J_{h,n+r+1} - J_{h,n-r+1}) \\ + j(J_{h,n+r+2} - J_{h,n-r+2}) + k(J_{h,n+r+3} - J_{h,n-r+3}).$$

Using the property (iv) proved in Lemma 2.3, we have

$$\begin{aligned} (h+1)(JQ_{h,n+r} - JQ_{h,n-r}) &= h^{n-r}(h^{2r} - 1) \\ &\quad + ih^{n+1-r}(h^{2r} - 1) \\ &\quad + jh^{n+2-r}(h^{2r} - 1) \\ &\quad + kh^{n+3-r}(h^{2r} - 1) \\ &= h^{n-r}(h^{2r} - 1)(1 + hi + h^2j + h^3k). \end{aligned}$$

(v) By definition,

$$N(JQ_{h,n}) = J_{h,n}^2 + J_{h,n+1}^2 + J_{h,n+2}^2 + J_{h,n+3}^2.$$

On the other hand, by Lemma 2.1 we have

$$\begin{aligned} &(h+1)^2(J_{h,n}^2 + J_{h,n+1}^2 + J_{h,n+2}^2 + J_{h,n+3}^2) \\ &= (h^n - (-1)^n)^2 + (h^{n+1} - (-1)^{n+1})^2 + (h^{n+2} - (-1)^{n+2})^2 + (h^{n+3} - (-1)^{n+3})^2 \\ &= h^{2n} - 2(-1)^n h^n + (-1)^{2n} + h^{2(n+1)} - 2(-1)^{n+1} h^{n+1} + (-1)^{2(n+1)} + h^{2(n+2)} \\ &\quad - 2(-1)^{n+2} h^{n+2} + (-1)^{2(n+2)} + h^{2(n+3)} - 2(-1)^{n+3} h^{n+3} + (-1)^{2(n+3)} \\ &= h^{2n} - 2(-1)^n h^n + h^{2n+2} + 2(-1)^n h^{n+1} + h^{2n+4} - 2(-1)^n h^{n+2} + h^{2n+6} \\ &\quad + 2(-1)^n h^{n+3} + 4 \\ &= h^{2n}(1 + h^2 + h^4 + h^6) + 2(-1)^{n-1} h^n(1 - h + h^2 - h^3) + 4 \\ &= h^{2n}(1 + h^2)(1 + h^4) + 2(-1)^n h^n(h - 1)(1 + h^2) + 4. \end{aligned}$$

□

Other interesting relations referred to Jacobstal-Lucas quaternions had been obtained by A. Szydal-Liana and I. Włoch see [26]. Most of them had been resumed and by F. T. Aydin and S. Yüce (see [2]) who also gave the Binet's Formulas and the Cassini's Identity for Jacobstal quaternions and Jacobstal-Lucas quaternions.

Theorem 3.2. [see [2, Equations (43)–(47)] and [26, Theorem 2]] Let $n \geq 1$ and $r \geq 1$. Then:

$$i) \quad JLQ_{n+1} + JLQ_n = 32^n(1 + 2i + 2^2j + 2^3k).$$

$$ii) \quad JLQ_{n+1} - JLQ_n = 2^n(1 + 2i + 4j + 8k) + 2(-1)^n(1 - i + j - k).$$

$$iii) \quad JLQ_{n+r} + JLQ_{n-r} = [2^{n-r}(2^{2r} + 1)(1 + 2i + 2^2j + 2^3k) - 2(-1)^{n+r}(1 - i + j - k)].$$

$$iv) \quad JLQ_{n+r} - JLQ_{n-r} = 2^{n-r}(2^{2r} - 1)(1 + 2i + 2^2j + 2^3k).$$

$$v) \quad N(JLQ_n) = 85 \cdot 2^{2n} - 10 \cdot 2^n(-1)^n + 4.$$

The next result, that is referred to \bar{h} -Jacobstal-Lucas quaternions, will extend the above one: it can be obtained just replacing $h = 2$ in the following statement.

Theorem 3.3. *Let $h \neq -1$ be a real number, and let $(JLQ_{h,n})_n$ be the \bar{h} -Jacobstal-Lucas quaternions sequence. For every positive integers n and r , we have:*

$$\begin{aligned}
 i) \quad JLQ_{h,n+1} + JLQ_{h,n} &= 3h^n(1 + hi + h^2j + h^3k). \\
 ii) \quad JLQ_{h,n+1} - JLQ_{h,n} &= \frac{3h^n(h-1)}{h+1}(1 + hi + h^2j + h^3k) \\
 &\quad - \frac{2}{h+1}(2h-1)(-1)^n(1-i+j-k). \\
 iii) \quad JLQ_{h,n+r} + JLQ_{h,n-r} &= \frac{1}{h+1}[3h^{n-r}(h^{2r}+1)(1 + hi + h^2j + h^3k) \\
 &\quad + 2(2h-1)(-1)^{n+r}(1-i+j-k)]. \\
 iv) \quad JLQ_{h,n+r} - JLQ_{h,n-r} &= \frac{3}{h+1}h^{n-r}(h^{2r}-1)(1 + hi + h^2j + h^3k). \\
 v) \quad N(JLQ_{h,n}) &= \frac{1}{(h+1)^2}[4(2h-1)^2 + 9h^{2n}(1 + h^2 + h^4 + h^6) \\
 &\quad + 2 \cdot 3h^n(2h-1)(-1)^n(1-h+h^2-h^3)].
 \end{aligned}$$

Proof. (i) By definition,

$$\begin{aligned}
 JLQ_{h,n+1} + JLQ_{h,n} &= (j_{h,n+1} + j_{h,n}) + i(j_{h,n+2} + j_{h,n+1}) \\
 &\quad + j(j_{h,n+3} + j_{h,n+2}) + k(j_{h,n+4} + j_{h,n+3}).
 \end{aligned}$$

Using the property (i) proved in Lemma 2.3, we have

$$JLQ_{h,n+1} + JLQ_{h,n} = 3h^n(1 + hi + h^2j + h^3k).$$

(ii) By definition,

$$\begin{aligned}
 JLQ_{h,n+1} - JLQ_{h,n} &= (j_{h,n+1} - j_{h,n}) + i(j_{h,n+2} - j_{h,n+1}) \\
 &\quad + j(j_{h,n+3} - j_{h,n+2}) + k(j_{h,n+4} - j_{h,n+3}).
 \end{aligned}$$

Using the property (ii) proved in Lemma 2.3, we have

$$\begin{aligned}
 JLQ_{h,n+1} - JLQ_{h,n} &= \frac{1}{h+1}(3h^n(h-1) - 2(2h-1)(-1)^n) \\
 &\quad + \frac{1}{h+1}(3hh^n(h-1) + 2(2h-1)(-1)^n)i \\
 &\quad + \frac{1}{h+1}(3h^2h^n(h-1) - 2(2h-1)(-1)^n)j
 \end{aligned}$$

$$\begin{aligned} & + \frac{1}{h+1} (3h^3h^n(h-1) + 2(2h-1)(-1)^n)k \\ & = \frac{3h^n(h-1)}{h+1}(1+hi+h^2j+h^3k) - \frac{2}{h+1}(2h-1)(-1)^n(1-i+j-k). \end{aligned}$$

(iii) By definition,

$$\begin{aligned} JLQ_{h,n+r} + JLQ_{h,n-r} &= (j_{h,n+r} + j_{h,n-r}) + i(j_{h,n+r+1} + j_{h,n-r+1}) \\ & \quad + j(j_{h,n+r+2} + j_{h,n-r+2}) + k(j_{h,n+r+3} + j_{h,n-r+3}). \end{aligned}$$

Using the property (iii) proved in Lemma 2.3, we have

$$\begin{aligned} JLQ_{h,n+r} + JLQ_{h,n-r} &= \frac{1}{h+1} [3h^{n-r}(h^{2r} + 1) + 2(2h-1)(-1)^{n+r}] \\ & \quad + \frac{1}{h+1} [3h^{n+1-r}(h^{2r} + 1) + 2(2h-1)(-1)^{n+1+r}]i \\ & \quad + \frac{1}{h+1} [3h^{n+2-r}(h^{2r} + 1) + 2(2h-1)(-1)^{n+2+r}]j \\ & \quad + \frac{1}{h+1} [3h^{n+3-r}(h^{2r} + 1) + 2(2h-1)(-1)^{n+3+r}]k \\ &= \frac{1}{h+1} [3h^{n-r}(h^{2r} + 1) + 2(2h-1)(-1)^{n+r}] \\ & \quad + \frac{1}{h+1} [3hh^{n-r}(h^{2r} + 1) - 2(2h-1)(-1)^{n+r}]i \\ & \quad + \frac{1}{h+1} [3h^2h^{n-r}(h^{2r} + 1) + 2(2h-1)(-1)^{n+r}]j \\ & \quad + \frac{1}{h+1} [3h^3h^{n-r}(h^{2r} + 1) - 2(2h-1)(-1)^{n+r}]k \\ &= \frac{1}{h+1} [3h^{n-r}(h^{2r} + 1)(1 + hi + h^2j + h^3k) + 2(2h-1)(-1)^{n+r}(1 - i + j - k)]. \end{aligned}$$

(iv) By definition,

$$\begin{aligned} JLQ_{h,n+r} - JLQ_{h,n-r} &= (j_{h,n+r} - j_{h,n-r}) + i(j_{h,n+r+1} - j_{h,n-r+1}) \\ & \quad + j(j_{h,n+r+2} - j_{h,n-r+2}) + k(j_{h,n+r+3} - j_{h,n-r+3}). \end{aligned}$$

Using the property (iv) proved in Lemma 2.3, we have

$$\begin{aligned} JLQ_{h,n+r} - JLQ_{h,n-r} &= \frac{3h^{n-r}}{h+1}(h^{2r} - 1) \\ & \quad + \frac{3hh^{n-r}}{h+1}(h^{2r} - 1)i \\ & \quad + \frac{3h^2h^{n-r}}{h+1}(h^{2r} - 1)j \\ & \quad + \frac{3h^3h^{n-r}}{h+1}(h^{2r} - 1)k \\ &= \frac{3}{h+1} h^{n-r}(h^{2r} - 1)(1 + hi + h^2j + h^3k). \end{aligned}$$

(v) By definition, $N(JQ_{h,n}) = j_{h,n}^2 + j_{h,n+1}^2 + j_{h,n+2}^2 + j_{h,n+3}^2$. Using the Binet's Formula we have:

$$\begin{aligned} N(JQ_{h,n}) &= \left(\frac{1}{h+1} [3h^n + (2h-1)(-1)^n]\right)^2 + \left(\frac{1}{h+1} [3hh^n + (2h-1)(-1)^{n+1}]\right)^2 \\ & \quad + \left(\frac{1}{h+1} [3h^2h^n + (2h-1)(-1)^{n+2}]\right)^2 + \left(\frac{1}{h+1} [3h^3h^n + (2h-1)(-1)^{n+3}]\right)^2 \\ &= \frac{1}{(h+1)^2} \{ [3h^n + (2h-1)(-1)^n]^2 + [3hh^n + (2h-1)(-1)^{n+1}]^2 \\ & \quad + [3h^2h^n + (2h-1)(-1)^{n+2}]^2 + [3h^3h^n + (2h-1)(-1)^{n+3}]^2 \} \end{aligned}$$

$$= \frac{1}{(h+1)^2} [4(2h-1)^2 + 9h^{2n}(1+h^2+h^4+h^6) + 2 \cdot 3h^n(2h-1) \cdot (-1)^n(1-h+h^2-h^3)].$$

□

Now we give the matrix representation of \bar{h} -Jacobsthal and \bar{h} -Jacobsthal-Lucas quaternions. The following result extends Theorem 5 of [26], which can be obtained by next theorem just replacing $h = 2$.

Theorem 3.4. *Let $n \geq 1$ be integer and let $h \neq -1$ be a real number. Then*

$$\begin{pmatrix} JQ_{h,n} & JQ_{h,n-1} \\ JQ_{h,n+1} & JQ_{h,n} \end{pmatrix} = \begin{pmatrix} JQ_{h,1} & JQ_{h,0} \\ JQ_{h,2} & JQ_{h,1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ h & 0 \end{pmatrix}^{n-1}.$$

Proof. If $n = 1$ then the result is obvious. Let $n > 1$ and proceeding by induction assume that

$$\begin{pmatrix} JQ_{h,n-1} & JQ_{h,n-2} \\ JQ_{h,n} & JQ_{h,n-1} \end{pmatrix} = \begin{pmatrix} JQ_{h,1} & JQ_{h,0} \\ JQ_{h,2} & JQ_{h,1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ h & 0 \end{pmatrix}^{n-2}. \text{ It follows}$$

that:

$$\begin{aligned} & \begin{pmatrix} JQ_{h,1} & JQ_{h,0} \\ JQ_{h,2} & JQ_{h,1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ h & 0 \end{pmatrix}^{n-1} \\ &= \begin{pmatrix} JQ_{h,1} & JQ_{h,0} \\ JQ_{h,2} & JQ_{h,1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ h & 0 \end{pmatrix}^{n-2} \begin{pmatrix} 1 & 1 \\ h & 0 \end{pmatrix} \\ &= \begin{pmatrix} JQ_{h,n-1} & JQ_{h,n-2} \\ JQ_{h,n} & JQ_{h,n-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ h & 0 \end{pmatrix} \\ &= \begin{pmatrix} JQ_{h,n} & JQ_{h,n-1} \\ JQ_{h,n+1} & JQ_{h,n} \end{pmatrix}. \quad \square \end{aligned}$$

The final result extends Theorem 6 of [26], which can be obtained by next theorem just replacing $h = 2$.

Theorem 3.5. *Let $n \geq 1$ be integer and let $h \neq -1$ be a real number. Then*

$$\begin{aligned} JQ_{h,n}^2 - JQ_{h,n+1}JQ_{h,n-1} &= [(h^3 - h^2 + h + 1) - (h-1)^2(h+1)i \\ &\quad - (h^3 - 3h^2 + h - 1)j + ((h-1)(h+1)^2)k](-h)^{n-1} \\ &= (JQ_{h,1}^2 - JQ_{h,2}JQ_{h,0})(-h)^{n-1}. \end{aligned}$$

Proof. Let $JQ_{h,n} = J_{h,n} + iJ_{h,n+1} + jJ_{h,n+2} + kJ_{h,n+3}$,

$$JQ_{h,n+1} = J_{h,n+1} + iJ_{h,n+2} + jJ_{h,n+3} + kJ_{h,n+4},$$

$$JQ_{h,n-1} = J_{h,n-1} + iJ_{h,n} + jJ_{h,n+1} + kJ_{h,n+2}. \text{ It follows that}$$

$$JQ_{h,n}^2 - JQ_{h,n+1}JQ_{h,n-1} = J_{h,n}^2 - J_{h,n+1}^2 - J_{h,n+2}^2 - J_{h,n+3}^2 - J_{h,n+1}J_{h,n-1} \quad (12)$$

$$\begin{aligned}
 &+ J_{h,n+2}J_{h,n} + J_{h,n+3}J_{h,n+1} + J_{h,n+4}J_{h,n+2} \\
 &+ i(J_{h,n+1}J_{h,n} - J_{h,n+2}J_{h,n-1} - J_{h,n+3}J_{h,n+2} + J_{h,n+4}J_{h,n-1}) \\
 &+ j(2J_{h,n+2}J_{h,n} - J_{h,n+1}^2 + J_{h,n+2}^2 - J_{h,n+3}J_{h,n-1} - J_{h,n+4}J_{h,n}) \\
 &+ k(3J_{h,n}J_{h,n+3} - 2J_{h,n+1}J_{h,n+2} - J_{h,n+4}J_{h,n-1}).
 \end{aligned}$$

Now we will compute the real, i , j and k part of $JQ_{h,n}^2 - JQ_{h,n+1}JQ_{h,n-1}$.

- By Equation 12

the real part of $JQ_{h,n}^2 - JQ_{h,n+1}JQ_{h,n-1}$ is

$$\begin{aligned}
 &J_{h,n}^2 - J_{h,n+1}^2 - J_{h,n+2}^2 - J_{h,n+3}^2 - J_{h,n+1}J_{h,n-1} + J_{h,n+2}J_{h,n} + J_{h,n+3}J_{h,n+1} \\
 &+ J_{h,n+4}J_{h,n+2}.
 \end{aligned}$$

On the other hand, by Binet's Formula (see 2.1) we have:

$$\begin{aligned}
 &(h+1)^2(J_{h,n}^2 - J_{h,n+1}^2 - J_{h,n+2}^2 - J_{h,n+3}^2 - J_{h,n+1}J_{h,n-1} + J_{h,n+2}J_{h,n} \\
 &+ J_{h,n+3}J_{h,n+1} + J_{h,n+4}J_{h,n+2}) \\
 &= h^{2n} - 2(-1)^n h^n + (-1)^{2n} - [h^{2(n+1)} - 2(-1)^{n+1} h^{n+1} + (-1)^{2(n+1)}] \\
 &\quad - [h^{2(n+2)} - 2(-1)^{n+2} h^{n+2} + (-1)^{2(n+2)}] \\
 &\quad - [h^{2(n+3)} - 2(-1)^{n+3} h^{n+3} + (-1)^{2(n+3)}] \\
 &\quad - [h^{2n} - (-1)^{n-1} h^{n+1} - (-1)^{n+1} h^{n-1} + (-1)^{2n}] + h^{2n+2} - (-1)^n h^{n+2} \\
 &\quad - (-1)^{n+2} h^n + (-1)^{2n+2} + h^{2n+4} - (-1)^{n+1} h^{n+3} - (-1)^{n+3} h^{n+1} \\
 &\quad + (-1)^{2n+4} + h^{2n+6} - (-1)^{n+2} h^{n+4} - (-1)^{n+4} h^{n+2} + (-1)^{2n+6} \\
 &= -(-1)^n h^{n+4} - (-1)^n h^{n+3} - 2(-1)^n h^{n+1} - 3(-1)^n h^n - (-1)^n h^{n-1} \\
 &= (-h)^{n-1} (h+1)^2 (h^3 - h^2 + h + 1).
 \end{aligned}$$

Therefore the real part of $JQ_{h,n}^2 - JQ_{h,n+1}JQ_{h,n-1}$ is:

$$(-h)^{n-1} (h^3 - h^2 + h + 1).$$

- The i -part of $JQ_{h,n}^2 - JQ_{h,n+1}JQ_{h,n-1}$ is

$$(J_{h,n+1}J_{h,n} - J_{h,n+2}J_{h,n-1} - J_{h,n+3}J_{h,n+2} + J_{h,n+4}J_{h,n+1}).$$

On the other hand by Binet's Formula (see 2.1) we have:

$$\begin{aligned}
 &(h+1)^2(J_{h,n+1}J_{h,n} - J_{h,n+2}J_{h,n-1} - J_{h,n+3}J_{h,n+2} + J_{h,n+4}J_{h,n+1}) \\
 &= h^{2n+1} - (-1)^n h^{n+1} - (-1)^{n+1} h^n + (-1)^{2n+1} \\
 &\quad - (h^{2n+1} - (-1)^{n-1} h^{n+2} - (-1)^{n+2} h^{n-1} + (-1)^{2n+1}) \\
 &\quad - (h^{2n+5} - (-1)^{n+2} h^{n+3} - (-1)^{n+3} h^{n+2} + (-1)^{2n+5})
 \end{aligned}$$

$$\begin{aligned}
 & + h^{2n+5} - (-1)^{n+1}h^{n+4} - (-1)^{n+4}h^{n+1} + (-1)^{2n+5} \\
 = & -(-1)^n h^{n+1} - (-1)^{n+1}h^n - (-(-1)^{n-1}h^{n+2} - (-1)^{n+2}h^{n-1}) \\
 & -(-(-1)^{n+2}h^{n+3} - (-1)^{n+3}h^{n+2}) - (-1)^{n+1}h^{n+4} - (-1)^{n+4}h^{n+1} \\
 = & (-1)^{n-1}h^{n+1} - (-1)^{n-1}h^n + (-1)^{n-1}h^{n+2} - (-1)^{n-1}h^{n-1} \\
 & - (-1)^{n-1}h^{n+3} + (-1)^{n-1}h^{n+2} - (-1)^{n-1}h^{n+4} + (-1)^{n-1}h^{n+1} \\
 = & (-1)^{n-1}(2h^{n+1} - h^n + 2h^{n+2} - h^{n-1} - h^{n+3} - h^{n+4}) \\
 & = (-h)^{n-1}(2h^2 - h + 2h^3 - 1 - h^4 - h^5) \\
 & = -(-h)^{n-1}(h-1)^2(h+1)^3.
 \end{aligned}$$

Therefore the i -part of $JQ_{h,n}^2 - JQ_{h,n+1}JQ_{h,n-1}$ is:
 $-(-h)^{n-1}(h-1)^2(h+1)$.

- Similarly, it can be checked that the j -part of $JQ_{h,n}^2 - JQ_{h,n+1}JQ_{h,n-1}$ is

$$-(-h)^{n-1}(h^3 - 3h^2 + h - 1).$$

- and that the k -part of $JQ_{h,n}^2 - JQ_{h,n+1}JQ_{h,n-1}$ is

$$(-h)^{n-1}(h-1)(h+1)^2.$$

□

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